CONSERVATIVE AND SEMICONSERVATIVE RANDOM WALKS: RECURRENCE AND TRANSIENCE

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Abstract. In the present paper we define conservative and semiconservative random walks in $\mathbb{Z}^d$ and study different families of random walks. The family of symmetric random walks is one of the families of conservative random walks, and simple (Pólya) random walks are their representatives. The classification of random walks given in the present paper enables us to provide a new approach to random walks in $\mathbb{Z}^d$ by reduction to birth-and-death processes. We construct nontrivial examples of recurrent random walks in $\mathbb{Z}^d$ for any $d \geq 3$ and transient random walks in $\mathbb{Z}^2$.

1. Introduction

It is well-known that the simple random walk (which is also called Pólya random walk) of dimension one or two is recurrent, i.e. being started from the origin, it returns to the original point infinitely many times. Pólya random walks of dimension higher than two are transient. These facts were originally established in 1921 by Pólya [12], and nowadays there are a number of new proofs and extensions of the Pólya theorem (e.g. [1], [3], [5], [8], [9], [11], and others).

Although the aforementioned results are well-known and have the distinguished history, the fact that random walks of dimension higher than two are not recurrent, while those of dimensions one and two are, remains mysterious and lying beyond the intuitive understanding. The further extensions of the Pólya theorem are chiefly based on application of the majorization (coupling) techniques of electric networks (e.g. [8]) or general methods using probability inequalities (e.g. [2] or [4], Lemma 4.2.5). They extend the result for the random walks of the same dimension to address the question whether or not a modified random walk of dimension two is recurrent or that of dimension three or higher is transient. As well, broad extensions to Pólya's theorem may be obtained by the method of Lyapunov functions, dating back to Lamperti [7].

We suggest another approach for classification of random walks. The main idea of our approach is to establish connection between random walks and birth-and-death processes. On the basis of this connection, we define new classes of random walks, called *conservative* and *semiconservative*. With the aid of this classification we are able to study the new cases, that was impossible in the frameworks of the earlier methods.

Our attention in the present paper is restricted by the random walks $\mathbf{S}_t = \big(S_t^{(1)}, S_t^{(2)}, \ldots, S_t^{(d)}\big)$ in $\mathbb{Z}^d$ ($t = 0, 1, \ldots$ is a discrete time parameter) satisfying the

1991 *Mathematics Subject Classification.* 60G50; 60J80; 60C05; 60K25.

Key words and phrases. Multi-dimensional random walks; birth-and-death process; Markovian single-server queues.

recurrence relations

$$\mathbf{S}_0 = \mathbf{0}, \tag{1.1}$$

$$\mathbf{S}_t = \mathbf{S}_{t-1} + \mathbf{e}_t,\ t \geq 1, \tag{1.2}$$

where $\mathbf{0}$ is the d-dimensional vector of zeros, and the random vector $\mathbf{e}_t$ takes the values $\pm\mathbf{1}_i$, $i = 1, 2, \ldots, d$, and $\mathbf{0}$.

Let $\mathcal{A}$ be a set of (vector-valued) parameters. Then, the triple $\{\mathbf{S}_t, \mathcal{A}, d\}$ is said to specify a family of random walks. Let $a \in \mathcal{A}$. Then, $\mathbf{S}_t(a, d)$ is a random walk that belongs to the family of random walks $\{\mathbf{S}_t, \mathcal{A}, d\}$.

The random walks that are studied in this paper are originated by the following three models.

Model 1. For random walks defined by (1.1) and (1.2), let the vector $\mathbf{e}_t$ be one of the $2d$ randomly chosen vectors $\{\pm\mathbf{1}_i, i = 1, 2, \ldots, d\}$ independently of the history and each other as follows. The probability that the vector $\mathbf{1}_i$ will be chosen, which is the same for the vector $(-\mathbf{1}_i)$, is equal to $\alpha_i > 0$, and $2\sum_{i=1}^d \alpha_i = 1$. Then the family $\{\mathbf{S}_t, \mathcal{A}, d\}$ is specified by the set of vectors $\big(\alpha_1, \alpha_2, \ldots, \alpha_d\big) \in \mathcal{A}$.

The random walks of Model 1 are called *symmetric* random walks, while the title *simple* or *simple symmetric* is related to Pólya random walks, which is one of the representatives of the symmetric random walks family.

Model 2. Assume now, that the vector $\mathbf{e}_t$ is specified as follows. If $S_{t-1}^{(i)} = 0$, then the vectors $\mathbf{1}_i$ and $(-\mathbf{1}_i)$ are chosen with equal probability α_i each, where $2\sum_{i=1}^d \alpha_i = 1$. If $S_{t-1}^{(i)} \neq 0$, then the probability to be chosen for each of the vectors $\mathbf{1}_i$ and $(-\mathbf{1}_i)$ is $\alpha_i - \delta_i/2 > 0$, where $\delta_i > 0$. Then, the vector $\mathbf{0}$ is chosen with the complementary probability. For instance, if $\mathbf{S}_{t-1} = (1, 3, -1)$, then the probability for vector $\mathbf{0}$ to be chosen is $(\delta_1 + \delta_2 + \delta_3)$. But if $\mathbf{S}_{t-1} = (0, 2, 0)$, then the probability for vector $\mathbf{0}$ to be chosen is δ_2. The family of random walks in this model is specified by the sets of vectors $\big(\alpha_1, \alpha_2, \ldots, \alpha_d\big)$ and $\big(\delta_1, \delta_2, \ldots, \delta_d\big)$, where the components of these vectors satisfy the inequality

$$1 \ \leq \frac{2\alpha_i}{2\alpha_i - \delta_i} \ < \ \infty.$$

Model 3. Assume now, that the vector $\mathbf{e}_t$ is specified as follows. If $S_{t-1}^{(i)} \neq 0$, then the vectors $\mathbf{1}_i$ and $(-\mathbf{1}_i)$ are chosen with equal probability α_i each, where $2\sum_{i=1}^d \alpha_i = 1$. If $S_{t-1}^{(i)} = 0$, then the probability to be chosen for each of the vectors $\mathbf{1}_i$ and $(-\mathbf{1}_i)$ is $\alpha_i - \delta_i/2 > 0$, where $\delta_i > 0$. Then, the vector $\mathbf{0}$ is chosen with the complementary probability. As in Model 2, the family of random walks is specified by the sets of vectors $\big(\alpha_1, \alpha_2, \ldots, \alpha_d\big)$ and $\big(\delta_1, \delta_2, \ldots, \delta_d\big)$, and the components of these vectors satisfy the inequalities

$$0 \ < \ \frac{2\alpha_i - \delta_i}{2\alpha_i} \ \leq \ 1.$$

In the case of Models 2 and 3, the set $\mathcal{A}$ is the sets of vectors $\big(\alpha_1, \alpha_2, \ldots, \alpha_d\big)$ and $\big(\delta_1, \delta_2, \ldots, \delta_d\big)$. That is, an element $a \in \mathcal{A}$ is the vector of dimension $2d$. It is easy to see that Model 1 is particular to both of Models 2 and 3 when $\delta_i = 0$, $i = 1, 2, \ldots, d$. All the three models could be amalgamated to a common model, the study of which can be further extended. For the purpose of the present paper, however, it is profitable to start from Models 1, 2 and 3 in order to initiate the study of more general models in a gradual way.

Let $\mathbf{n} = \big(n^{(1)}, n^{(2)}, \ldots, n^{(d)}\big)$ be a vector in $\mathbb{Z}^d$. Its norm is defined by

$$\|\mathbf{n}\| = \sum_{i=1}^{d} \big|n^{(i)}\big|.$$

By the sequence of *active* time instants $t_1, t_2, \ldots, t_j, \ldots$, we mean

$$t_1 = \inf\{t > 0 : \|\mathbf{S}_t\| > 0\}, \quad t_2 = \inf\{t > t_1 : \|\mathbf{S}_t\| \neq \|\mathbf{S}_{t_1}\|\}, \ldots,$$
$$t_j = \inf\{t > t_{j-1} : \|\mathbf{S}_t\| \neq \|\mathbf{S}_{t_{j-1}}\|\}, \ldots$$

In other words, the active time instants are the times when a random walk changes its state. Note, that for the family of random walks in Model 1, all the time instants $t = 1, 2, \ldots,$ are active.

By sequence of *up-crossing* time instants t'_1, t'_2,…, t'_j,…, we mean

$$t'_1 = t_1, \quad t'_2 = \inf\{t > t'_1 : \|\mathbf{S}_t\| = \|\mathbf{S}_{t-1}\| + 1\}, \ldots,$$
$$t'_j = \inf\{t > t'_{j-1} : \|\mathbf{S}_t\| = \|\mathbf{S}_{t-1}\| + 1\}, \ldots$$

In the following considerations, without loss of generality we assume that the time intervals between active time instants are exponentially distributed.

Specifically, with any state $\mathbf{n} \in \mathbb{Z}^d$ we associate $2d$ independent Poisson processes. As a random walk is in state $\mathbf{n}$, these Poisson processes define the direction of the following movement of that random walk. For instance, in the case of Model 1, for any state $\mathbf{n} \in \mathbb{Z}^d$ the Poisson processes have the rates

$$\alpha_1, \alpha_1, \alpha_2, \alpha_2, \ldots, \underbrace{\alpha_i}_{+\mathbf{1}_i}, \underbrace{\alpha_i}_{-\mathbf{1}_i}, \ldots, \alpha_d, \alpha_d, \tag{1.3}$$

where the notation $\underbrace{\alpha_i}_{+\mathbf{1}_i}$ indicates that the Poisson process with rate α_i is associated with the direction $\mathbf{1}_i$, and, respectively, the notation $\underbrace{\alpha_i}_{-\mathbf{1}_i}$ indicates that the Poisson process with rate α_i is associated with the direction $(-\mathbf{1}_i)$. So, in (1.3) the rates with odd order number in the row are associated with "positive" unit direction, while the rates with even order number with "negative". Then, the time between consecutive jumps is exponentially distributed with mean $2\sum_{i=1}^{d}\alpha_i = 1$, and the direction of the jump is associated with the location of the minimum of the $2d$ exponentially distributed "inter-jump" times associated with the aforementioned Poisson processes.

In the case of Models 2 or 3, the rates of Poisson processes are state dependent. If in the primary scale $\mathsf{E}(t_{j+1} - t_j) = \nu$, then the length $t_{j+1} - t_j$ in the new scale is to be exponentially distributed with mean ν.

Let $P_{t_j}^{(a)}(n)$, $n \geq 0$, $a \in \mathcal{A}$, denote the transition probability

$$P_{t_j}^{(a,d)}(n) = \mathsf{P}\big\{\|\mathbf{S}_{t_{j+1}}(a,d)\| = n + 1 \;\big|\; \|\mathbf{S}_{t_j}(a,d)\| = n\big\}.$$

Then,

$$p^{(a,d)}(n) = \lim_{j\to\infty} P_{t_j}^{(a,d)}(n) \tag{1.4}$$

for all $n \geq 0$ and $a \in \mathcal{A}$ is the notation for this limiting probability.

Definition 1.1. The family of random walks $\{\mathbf{S}_t, \mathcal{A}, d\}$ is called *conservative* (or $(\mathcal{A}, d)$-*conservative*), if for any $a_1 \in \mathcal{A}$ and $a_2 \in \mathcal{A}$ and all $n \geq 0$

$$p^{(a_1,d)}(n) = p^{(a_2,d)}(n) \equiv p^{(d)}(n),$$

and *semiconservative* ($(\mathcal{A}, d)$-*semiconservative*) if there exists $a^* \in \mathcal{A}$ such that for all $a \in \mathcal{A}$ and $n \geq 0$ either

$$p^{(a^*,d)}(n) \leq p^{(a,d)}(n),$$

or

$$p^{(a^*,d)}(n) \geq p^{(a,d)}(n).$$

The rest of the paper is organized as follows. In Section 2, we study the family of symmetric random walks of Model 1 and prove that it is $(\mathcal{A}, d)$-conservative, and hence the limit in (1.4) describes a unique sequence $p^{(a,d)}(n) \equiv p^{(d)}(n)$ for all $a \in \mathcal{A}$ and any d. The analysis in Section 3 is related to the families of random walks of Models 2 and 3. It is shown that the families of random walks of these models are semiconservative. In Section 4, we study recurrent and transient random walks. We prove that all previously considered Models 1, 2 and 3 are recurrent for $d \leq 2$ and transient for $d \geq 3$. We extend the results for a possibly most general state-dependent model (Model 5). In the same section, we give non-trivial examples of transient two-dimensional random walks and recurrent d-dimensional random walks for $d \geq 3$ (which reduce to the systems of specifically defined independent null-recurrent birth-and-death processes). In Section 5, we discuss the results and conclude the paper.

2. Symmetric random walks

In this section we study symmetric random walks for Model 1. Let $\{\mathbf{S}_t, \mathcal{A}, d\}$ denote a family of symmetric random walks in $\mathbb{Z}^d$, and let $\mathbf{S}_t(a, d)$ be a random walk, $a = (\alpha_1, \alpha_2, \ldots, \alpha_d)$.

Let $\varphi_t(a, d) = \|\mathbf{S}_t(a, d)\|$. The functional $\varphi_t(a, d)$ satisfies the following recurrence relations

$$\varphi_0(a, d) = 0, \tag{2.1}$$

$$\varphi_t(a, d) = |\varphi_{t-1}(a, d) + X_t(\varphi_{t-1}(a), a, d)|, \quad t \geq 1, \tag{2.2}$$

where

$$X_t(n, a, d) = \begin{cases} +1, & \text{with probability} \quad P_t^{(a,d)}(n), \\ -1, & \text{with probability} \quad Q_t^{(a,d)}(n) = 1 - P_t^{(a,d)}(n). \end{cases} \tag{2.3}$$

Let $BD(\gamma, d)$ denote the family of birth-and-death processes, $\gamma > 0$, $d \geq 1$ (d is the integer-valued parameter associated with the dimension of random walks), with the birth and death parameters $\lambda_n(\gamma, d)$ and $\mu_n(\gamma, d)$, respectively, where

$$\lambda_n(\gamma, d) = \sum_{i=1}^{d} i\gamma^i \binom{d}{i}\binom{n-1}{i-1} + \gamma\sum_{i=1}^{d-1}(d-i)\gamma^i\binom{d}{i}\binom{n-1}{i-1},$$

and

$$\mu_n(\gamma, d) = \sum_{i=1}^{d} i\gamma^i\binom{d}{i}\binom{n-1}{i-1},$$

$n \geq 1$. (The parameter $\lambda_0(\gamma, d) > 0$ is not used and hence can be taken arbitrarily.)

Notice that with $d = 1$ the family $BD(\gamma, 1)$ is trivial, since $\lambda_n(\gamma, 1) = \mu_n(\gamma, 1)$. Therefore, in the sequel the only case $d \geq 2$ is considered.

We prove the following result.

Theorem 2.1. *The family of random walks* $\{\mathbf{S}_t, \mathcal{A}, d\}$ *is* $(\mathcal{A}, d)$*-conservative. It is characterized by the family of birth-and-death processes* $BD(2, d)$*. Specifically,*

$$p^{(a,d)}(n) \equiv p^{(d)}(n) = \frac{\lambda_n(2, d)}{\lambda_n(2, d) + \mu_n(2, d)}$$

for all $a \in \mathcal{A}$ *and* $d \geq 2$.

Proof. For convenience, the arguments a and d in the following notation for $\mathbf{S}_t(a, d)$ is omitted, since in this proof we deal with a unique random walk. Along with the original random walk $\mathbf{S}_t$ in $\mathbb{Z}^d$, one can consider the reflected random walk $\breve{\mathbf{S}}_t = \left(\breve{S}_t^{(1)}, \breve{S}_t^{(2)}, \ldots, \breve{S}_t^{(d)}\right)$ in $\mathbb{Z}_+^d$ defined as

$$\breve{\mathbf{S}}_0 = \mathbf{0}, \tag{2.4}$$

$$\breve{\mathbf{S}}_t = \breve{\mathbf{S}}_{t-1} + \mathbf{r}_t, \quad t \geq 1, \tag{2.5}$$

where

$$\mathbf{r}_t = \begin{cases} \mathbf{e}_t, & \text{if } \breve{S}_{t-1}^{(i)} + e_t^{(i)} \geq 0 \text{ for all } i = 1, 2, \ldots, d, \\ -\mathbf{e}_t, & \text{if } \breve{S}_{t-1}^{(i)} + e_t^{(i)} = -1 \text{ for a certain } i = 1, 2, \ldots, d, \end{cases}$$

and the vector $\mathbf{e}_t = \left(e_t^{(1)}, e_t^{(2)}, \ldots, e_t^{(d)}\right)$ is the vector that was defined earlier for the random walk $\mathbf{S}_t$ in (1.1) and (1.2).

Observing (2.4), (2.5) and their comparison with (1.1), (1.2) enables us to conclude that $\|\breve{\mathbf{S}}_t\|$ and $\|\mathbf{S}_t\|$ coincide in distribution. So, one can restrict ourselves by modeling the random walk $\breve{\mathbf{S}}_t$. The last one is described by the d independent queueing processes as follows. Assume that arrivals in the ith queueing system are Poisson with rate α_i, and service times are exponentially distributed with the same rate α_i. If a system becomes free, it is switched for a special service with the same rate α_i. This service is *negative*, and it results in a new customer in the queue. If during a *negative service* a new arrival occurs, the negative service remains unfinished and not resumed.

The negative service models the reflection at zero and in fact implies the state-dependent arrival rate, which becomes equal to $2\alpha_i$ at the moment when the system is empty. It is associated with the situation, when an original one-dimensional random walk reaches zero at some time moment s, and at the next time moment $s + 1$ it must take one of values ± 1 that corresponds to value $+1$ for an one-dimensional random walk *reflected at zero*.

To establish necessary properties of the d independent queueing systems, assume first that the number of waiting places in each of the queueing systems is N, where N is taken large enough, such that for a given d-dimensional vectors $\mathbf{n} \in \mathbb{Z}_+^d$, we have $\|\mathbf{n}\| < N$. The assumption on limited number of waiting places means that an arriving customer, who meets N customers in the system, is lost. Let $P_N(\mathbf{n})$ denote the stationary probability to be in state $\mathbf{n}$ immediately before an arrival of a customer in one of the d queueing systems. Following the PASTA property [13], the stationary probabilities can be derived on the basis of the backward Chapman-Kolmogorov equations for a continuous Markov process (the random walk $\breve{\mathbf{S}}_t$ is reckoned to be extended to the continuous time process). Calculate first the stationary probability for a single queueing system. Denote by $P_N^{(i)}(n)$ the stationary probability to be in state n in the ith queueing system. From the balance equations, we obtain $P_N^{(i)}(n+1) = P_N^{(i)}(n)$ for $n \geq 1$, and $P_N(1) = 2P_N(0)$. So, we arrive at

$$P_N^{(i)}(n) = \begin{cases} \frac{2}{2N+1}, & \text{for } 1 \leq n \leq N, \\ \frac{1}{2N+1}, & \text{for } n = 0. \end{cases} \tag{2.6}$$

Notice that the stationary probabilities do not depend on the rate α_i. Since the queueing systems are independent, we arrive at the product form solution

$$P_N(\mathbf{n}) = \prod_{i=1}^{d} P_N^{(i)}\left(n^{(i)}\right). \tag{2.7}$$

Substituting (2.6) into (2.7) we may obtain the exact form solution. For any vector $\mathbf{n} \in \mathbb{Z}_+^d$, let $d_0(\mathbf{n})$ denote the number of zero components in the presentation of the vector $\mathbf{n}$. For example, vector (1,0,2,0) contains two zero components. Then,

$$P_N(\mathbf{n}) = 2^{d-d_0(\mathbf{n})}\frac{1}{(2N+1)^d} \tag{2.8}$$

is the required formula for the stationary probability, and for two arbitrary states $\mathbf{n}_1$ and $\mathbf{n}_2$, the ratio

$$\frac{P_N(\mathbf{n}_1)}{P_N(\mathbf{n}_2)} = 2^{d_0(\mathbf{n}_2)-d_0(\mathbf{n}_1)}$$

does not depend on N. Hence,

$$\lim_{N\to\infty}\frac{P_N(\mathbf{n}_1)}{P_N(\mathbf{n}_2)} = 2^{d_0(\mathbf{n}_2)-d_0(\mathbf{n}_1)}. \tag{2.9}$$

Next, denote

$$\Pi_\infty(\mathbf{n}_1, \mathbf{n}_2) = \lim_{j\to\infty}\frac{\mathsf{P}\left\{\breve{\mathbf{S}}_{t_j'-1} = \mathbf{n}_1\right\}}{\mathsf{P}\left\{\breve{\mathbf{S}}_{t_j'-1} = \mathbf{n}_2\right\}}. \tag{2.10}$$

Here, in (2.10), the subindex $t_j' - 1$ denotes the time instant preceding the jth up-crossing time instant t_j'. As well, by the product rule and the PASTA property

$$\Pi_\infty(\mathbf{n}_1, \mathbf{n}_2) = \prod_{i=1}^{d}\Pi_\infty^{(i)}\left(n_1^{(i)}, n_2^{(i)}\right) = \prod_{i=1}^{d}\lim_{j\to\infty}\frac{\mathsf{P}\left\{\breve{S}_{t_j'-1}^{(i)} = n_1^{(i)}\right\}}{\mathsf{P}\left\{\breve{S}_{t_j'-1}^{(i)} = n_2^{(i)}\right\}}.$$

Note, that

$$\sum_{n=0}^{\infty}\lim_{j\to\infty}\mathsf{P}\left\{\breve{S}_{t_j'-1}^{(i)} = n\right\} = \lim_{j\to\infty}\sum_{n=0}^{\infty}\mathsf{P}\left\{\breve{S}_{t_j'-1}^{(i)} = n\right\} = 1, \quad i = 1, 2, \ldots, d,$$

and on the basis of the Chapman-Kolmogorov equations for large j we obtain

$$\mathsf{P}\left\{\breve{S}_{t_j'-1}^{(i)} = 1\right\} = 2\mathsf{P}\left\{\breve{S}_{t_j'-1}^{(i)} = 0\right\}[1 + o(1)],$$

and

$$\mathsf{P}\left\{\breve{S}_{t_j'-1}^{(i)} = n+1\right\} = \mathsf{P}\left\{\breve{S}_{t_j'-1}^{(i)} = n\right\}[1 + o(1)]$$

for $n\geq 1$. Then, for $n_1^{(i)}>n_2^{(i)}$, we obtain

$$\lim_{j\to\infty}\frac{\mathsf{P}\left\{\breve{S}^{(i)}_{t'_j-1}=n_1^{(i)}\right\}}{\mathsf{P}\left\{\breve{S}^{(i)}_{t'_j-1}=n_2^{(i)}\right\}}=\begin{cases}1, & \text{if } n_2^{(i)}>0,\\ 2, & \text{if } n_2^{(i)}=0.\end{cases} \tag{2.11}$$

Now, according to (2.9), (2.10) and (2.11), we arrive at

$$\Pi_\infty(\mathbf{n}_1,\mathbf{n}_2)=\lim_{N\to\infty}\frac{P_N(\mathbf{n}_1)}{P_N(\mathbf{n}_2)}=2^{d_0(\mathbf{n}_2)-d_0(\mathbf{n}_1)}. \tag{2.12}$$

Now, let $\mathcal{N}^+(n)$ denote the set of all vectors in $\mathbb{Z}_+^d$ having norm n. The total number of vectors having norm n in $\mathbb{Z}_+^d$ is

$$\sum_{i=1}^{d}\binom{d}{i}\binom{n-1}{i-1}. \tag{2.13}$$

Here and later on, the formula sums over i being the number of nonzero components of the vector.

Hence, denoting the stationary state probability to belong to the set $\mathcal{N}^+(n)$ by $P_N[\mathcal{N}^+(n)]$, from (2.8) we obtain

$$P_N[\mathcal{N}^+(n)]=\sum_{\mathbf{n}\in\mathcal{N}^+(n)}P_N(\mathbf{n})=\frac{1}{(2N+1)^d}\sum_{i=1}^{d}2^i\binom{d}{i}\binom{n-1}{i-1}. \tag{2.14}$$

Here, in the right-hand side of (2.14), the term $\sum_{i=1}^{d}2^i\binom{d}{i}\binom{n-1}{i-1}$ characterizes the total number of elements in $\mathbb{Z}^d$ having norm n.

Let $p_n(d)$ denote the transition probability from the set of states $\mathcal{N}^+(n)$ (level n) to the set of states $\mathcal{N}^+(n+1)$ (level $n+1$), and let $q_n(d)=1-p_n(d)$ denote the transition probability from the level n to the level $n-1$.

Our task now is to derive the formula for $p_n(d)$, and for this derivation we use combinatorial arguments.

The total number of vectors in the set $\mathcal{N}^+(n)$ is given by (2.13). Each vector contains d components. Hence, the total number of components in the set of vectors in $\mathcal{N}^+(n)$ is

$$d\sum_{i=1}^{d}\binom{d}{i}\binom{n-1}{i-1}. \tag{2.15}$$

Among them, the total number of zero components is

$$\sum_{i=1}^{d-1}(d-i)\binom{d}{i}\binom{n-1}{i-1},$$

and the total number of nonzero components in all the aforementioned vectors is

$$\sum_{i=1}^{d}i\binom{d}{i}\binom{n-1}{i-1}.$$

Based on (2.8) or (2.14), it will be proved below the following relationships

$$
\begin{aligned}
p_n(d) &= \frac{2\sum_{i=1}^{d-1}(d-i)2^i\binom{d}{i}\binom{n-1}{i-1}+\sum_{i=1}^{d}i2^i\binom{d}{i}\binom{n-1}{i-1}}{2d\sum_{i=1}^{d}2^i\binom{d}{i}\binom{n-1}{i-1}} \\
&= \frac{C_0(n,d)+C(n,d)}{C_0(n,d)+2C(n,d)},
\end{aligned}
\tag{2.16}
$$

and

$$
q_n(d)=\frac{C(n,d)}{C_0(n,d)+2C(n,d)},
$$

where

$$
C(n,d)=\sum_{i=1}^{d}i2^i\binom{d}{i}\binom{n-1}{i-1},
\tag{2.17}
$$

and

$$
C_0(n,d)=2\sum_{i=1}^{d-1}(d-i)2^i\binom{d}{i}\binom{n-1}{i-1}.
\tag{2.18}
$$

The plan of the proof is as follows. We first consider the case of $\alpha_i=1/(2d)$ for all $i=1,2,\ldots,d$ (Pólya random walk), and then develop the proof for the general situation.

Case of Pólya random walk. First, we build the sample space. The components of all vectors in $\mathcal{N}^+(n)$, the total number of which is given by (2.15) are not equally likely. According to (2.8), a nonzero component appears with two times higher probability than a zero component. To make the components equally likely, we are to extend the number of nonzero components by factor 2. Then the total number of equally likely components is to be equal to

$$
d\sum_{i=1}^{d}2^i\binom{d}{i}\binom{n-1}{i-1}.
\tag{2.19}
$$

Following (2.19), the sample space for level n contains

$$
2d\sum_{i=1}^{d}2^i\binom{d}{i}\binom{n-1}{i-1}=C_0(n,d)+2C(n,d)
$$

states characterizing the number of possible transitions of the vectors in $\mathbb{Z}^d$ having norm n, where $C(n,d)$ and $C_0(n,d)$ are given by (2.17) and (2.18). Specifically, $C_0(n,d)$ is the number of possible transitions associated with reflections at zero. The rest of all possible transitions of the sample space is $2C(n,d)$. Half of them characterize transitions from level n to $n+1$ and half from level n to $n-1$. Hence, the total number of transitions from level n to level $n+1$ is $C_0(n,d)+C(n,d)$, and we arrive at (2.16). So, in the case $\alpha_i=1/(2d)$, $i=1,2,\ldots,d$ relation (2.16) is explained.

General case. We prove now, that (2.16) remains true in the general case of $a=(\alpha_1,\alpha_2,\ldots,\alpha_d)$, in which the probability of a transition of the jth coordinate of a vector is

$$
r_j=\frac{\alpha_j}{\alpha_1+\alpha_2+\ldots+\alpha_d}.
$$

Since, the set of states has the symmetric structure, then the contribution of the different terms r_j, $j = 1, 2, \ldots, d$ changes the terms $C_0(n, d)$ and $C(n, d)$ proportionally. Indeed, consider the ith term of the sum in (2.17). It is associated with $i2^i\binom{d}{i}\binom{n-1}{i-1}$ elements. Assuming that the choice of the jth coordinate of the vector $\mathbf{n}$ has the weight r_j, it can be seen that the fraction of the terms (weights) r_j for different j is the same among the total $i2^i\binom{d}{i}\binom{n-1}{i-1}$ elements. That is, the terms r_j all are uniformly concentrated among $i2^i\binom{d}{i}\binom{n-1}{i-1}$ elements, and this is true for all i. In this case, instead of the term $C(n, d)$ we have the modified term (weighted sum) denoted by $\tilde{C}(n, d)$. Then, we write the relation

$$\tilde{C}(n, d) = \tilde{c}(n, d)C(n, d). \tag{2.20}$$

However, the similar arguments are valid for the ith terms of the sum in (2.18). Hence, denoting the corresponding weighted sum by $\tilde{C}_0(n, d)$, we have

$$\tilde{C}_0(n, d) = \tilde{c}(n, d)C_0(n, d) \tag{2.21}$$

with the same proportion coefficient $\tilde{c}(n, d)$. That is, instead of $C_0(n, d)$ and $C(n, d)$ we are to have $\tilde{C}(n, d)$ and $\tilde{C}_0(n, d)$ defined by (2.20) and (2.21), respectively. Hence, the constant $\tilde{c}(n, d)$ being presented both in numerator and denominator of (2.16) finally reduces, and the probabilities r_j, $j = 1, 2, \ldots, d$, have no influence on the parameters $p_n(d)$ and $q_n(d)$.

As we can see, these transition probabilities depend neither on N nor on parameters α_i, $i = 1, 2, \ldots, d$, but on d only. Hence, as N increases to infinity, the parameters $p_n(d)$ and $q_n(d)$ remain unchanged. This means that for any $a \in \mathcal{A}$, the limiting relation in (1.4) is satisfied, and with (2.16) the limiting birth-and-death process is $BD(2, d)$. Thus, the family of random walks $\{\mathbf{S}_t, \mathcal{A}, d\}$ is $(\mathcal{A}, d)$-conservative. □

3. State-dependent random walks of Models 2 and 3

In this section we prove the following results.

Theorem 3.1. *The family of random walks $\{\mathbf{S}_t, \mathcal{A}, d\}$ defined in Models 2 and 3 are semiconservative.*

Proof. The idea of the proof for each of Models 2 and 3 is similar to that given in the proof of Theorem 2.1 for Model 1. Consider first the family of random walks of Model 2. We are to consider the reflected version of the random walk (denoted by $\breve{\mathbf{S}}_t$) and model it as d state-dependent queueing systems, which according to their construction are independent of each other. Let $\mathbf{n} = \big(n^{(1)}, n^{(2)}, \ldots, n^{(d)}\big)$, $n^{(i)} \geq 0$, be a vector. Denote the state-dependent arrival and service rates for the ith queueing system, since they are equal, both by $\tilde{\beta}_i\big(n^{(i)}\big)$. That is, if $n^{(i)} \geq 1$, then $\tilde{\beta}_i\big(n^{(i)}\big) = \alpha_i - \delta_i/2$, $i = 1, 2, \ldots, d$. Otherwise, if $n^{(i)} = 0$, then the arrival rate $\tilde{\beta}_i(0) = 2\alpha_i$. The role of coefficient 2 (negative service doubles arrival rate) is explained in the proof of Theorem 2.1. As in the proof of Theorem 2.1 assume first that the number of waiting places is N, where N is taken larger than the norm of vector $\mathbf{n}$, that is, $N > \|\mathbf{n}\|$. Let $P_N(\mathbf{n})$ denote the stationary probability. (The details of the definition are the same as in the proof of Theorem 2.1.) To derive the expression for $P_N(\mathbf{n})$, consider first an ith queueing system, and denote by $P_N^{(i)}(n)$

the stationary probability to be in the state n. We have the following relationships. For $n \geq 1$, $P_N^{(i)}(n+1) = P_N^{(i)}(n)$, and $P_N^{(i)}(1) = [4\alpha_i/(2\alpha_i - \delta_i)]P_N^{(i)}(0)$. Then,

$$P_N^{(i)}(n) = \begin{cases} \frac{4\alpha_i/(2\alpha_i-\delta_i)}{N[4\alpha_i/(2\alpha_i-\delta_i)]+1}, & \text{for} \quad 1 \leq n \leq N, \\ \frac{1}{N[4\alpha_i/(2\alpha_i-\delta_i)]+1}, & \text{for} \quad n = 0, \end{cases} \tag{3.1}$$

and since the queueing systems are independent,

$$P_N(\mathbf{n}) = \prod_{i=1}^{d} P_N^{(i)}\left(n^{(i)}\right) = \prod_{i=1}^{d} \frac{[4\alpha_i/(2\alpha_i - \delta_i)]^{\min\{n^{(i)},1\}}}{4N\alpha_i/(2\alpha_i - \delta_i) + 1}. \tag{3.2}$$

Similarly to (2.14), the level n probability in given by

$$P_N\left[\mathcal{N}^+(n)\right] = \sum_{\mathbf{n}\in\mathcal{N}^+(n)} P_N(\mathbf{n}).$$

where $P_N(\mathbf{n})$ are given by (3.2). Denote the transition probabilities from the set of states $\mathcal{N}^+(n)$ to the set of states $\mathcal{N}^+(n+1)$ by $p_n(d, k_1, k_2, \ldots, k_d)$, where $k_i = 2\alpha_i/(2\alpha_i - \delta_i)$. The explicit representation for $p_n(d, k_1, k_2, \ldots, k_d)$ is cumbersome, but in the particular case where $\delta_i = 0$, $i = 1, 2, \ldots, d$, it coincides with (2.16). As δ_i increases, the transition probability $p_n(d, k_1, k_2, \ldots, k_d)$ must either increase of decrease in dependence of the value d. The straightforward analytical proof of this based on the induction in d takes much place. Instead, we prove a more particular statement. Let

$$\underline{k} = \min_{1\leq i\leq d} \frac{2\alpha_i}{2\alpha_i - \delta_i}$$

and

$$\overline{k} = \max_{1\leq i\leq d} \frac{2\alpha_i}{2\alpha_i - \delta_i}.$$

We prove the justice of the following chain of the inequalities

$$p_n(d, \underline{k}) \leq p_n(d, \underbrace{k, k, \ldots, k}_{d \text{ times}}) \leq p_n(d, \overline{k}), \tag{3.3}$$

assuming that $k_1 = k_2 = \ldots = k_d = k$, $\underline{k} \leq k \leq \overline{k}$, where, based on the arguments in the proof of Theorem 2.1 (see the explanation for relation (2.16)), the explicit representations for $p_n(d, \underbrace{k, k, \ldots, k}_{d \text{ times}})$ as well as the lower and upper bounds $p_n(d, \overline{k})$ and $p_n(d, \underline{k})$ are

$$p_n(d, \underbrace{k, k, \ldots, k}_{d \text{ times}}) = \frac{2k\sum_{i=1}^{d-1}(d-i)(2k)^i\binom{d}{i}\binom{n-1}{i-1} + \sum_{i=1}^{d} i(2k)^i\binom{d}{i}\binom{n-1}{i-1}}{2k\sum_{i=1}^{d-1}(d-i)(2k)^i\binom{d}{i}\binom{n-1}{i-1} + 2\sum_{i=1}^{d} i(2k)^i\binom{d}{i}\binom{n-1}{i-1}}, \tag{3.4}$$

$$p_n(d, \overline{k}) = \frac{2\overline{k}\sum_{i=1}^{d-1}(d-i)(2\overline{k})^i\binom{d}{i}\binom{n-1}{i-1} + \sum_{i=1}^{d} i(2\overline{k})^i\binom{d}{i}\binom{n-1}{i-1}}{2\overline{k}\sum_{i=1}^{d-1}(d-i)(2\overline{k})^i\binom{d}{i}\binom{n-1}{i-1} + 2\sum_{i=1}^{d} i(2\overline{k})^i\binom{d}{i}\binom{n-1}{i-1}}, \tag{3.5}$$

and

$$p_n(d, \underline{k}) = \frac{2\underline{k}\sum_{i=1}^{d-1}(d-i)(2\underline{k})^i\binom{d}{i}\binom{n-1}{i-1} + \sum_{i=1}^{d} i(2\underline{k})^i\binom{d}{i}\binom{n-1}{i-1}}{2\underline{k}\sum_{i=1}^{d-1}(d-i)(2\underline{k})^i\binom{d}{i}\binom{n-1}{i-1} + 2\sum_{i=1}^{d} i(2\underline{k})^i\binom{d}{i}\binom{n-1}{i-1}}. \tag{3.6}$$

Indeed, we are to prove that the derivative in k for the right-hand side of (3.4) is strictly positive. Indeed, following (3.4) we have the representation

$$p_n(d, \underbrace{k, k, \ldots, k}_{d \text{ times}}) = \frac{f(2k)+1}{f(2k)+2},$$

where

$$f(k) = \frac{\sum_{i=1}^{d-1}(d-i)k^{i+1}\binom{d}{i}\binom{n-1}{i-1}}{\sum_{i=1}^{d} ik^i\binom{d}{i}\binom{n-1}{i-1}}.$$

Hence, the task is to show that $f(k)$ is a decreasing function. Taking the derivative of $f(k)$, we are to show the inequality

$$\begin{aligned}(3.7)\qquad &\sum_{i=1}^{d} ik^i\binom{d}{i}\binom{n-1}{i-1}\sum_{j=2}^{d} j(d-j+1)k^{j-1}\binom{d}{j-1}\binom{n-1}{j-2}\\ &<\sum_{i=1}^{d} i^2k^{i-1}\binom{d}{i}\binom{n-1}{i-1}\sum_{j=2}^{d}(d-j+1)k^j\binom{d}{j-1}\binom{n-1}{j-2}.\end{aligned}$$

The justice of (3.7) follows from the fact that for $j+k=2i$, we have $jk \leq i^2$. Thus the chains of inequalities (3.3) is correct. This constitutes that the family of random walks of Model 2 is semiconservative, since the obtained results remain unchanged in the limit as N tends to infinity.

When $2\alpha_i/(2\alpha_i-\delta_i)=\overline{k}$ for all $i=1,2,\ldots,d$, the family of random walks is associated with $BD(2\overline{k},d)$, and when $2\alpha_i/(2\alpha_i-\delta_i)=\underline{k}$ for all $i=1,2,\ldots,d$, the family of random walks is associated with $BD(2\underline{k},d)$.

For the family of random walks of Model 3 the proof is similar. For $P_N(\mathbf{n})$ we obtain

$$P_N(\mathbf{n}) = \prod_{i=1}^{d}\frac{[(2\alpha_i-\delta_i)/\alpha_i]^{\min\{n^{(i)},1\}}}{N(2\alpha_i-\delta_i)/\alpha_i+1},$$

$$P_N\left[\mathcal{N}^+(n)\right] = \sum_{\mathbf{n}\in\mathcal{N}^+(n)} P_N(\mathbf{n}),$$

and similarly to (3.3) the chain of inequalities

$$p_n(d,\underline{k}) \leq p_n(d, \underbrace{k, k, \ldots, k}_{d \text{ times}}) \leq p_n(d,\overline{k}), \quad \underline{k}\leq k\leq\overline{k},$$

with (3.4), (3.5) and (3.6), where $\underline{k}$ and $\overline{k}$ are now redefined as

$$\underline{k} = \min_{1\leq i\leq d}\frac{2\alpha_i-\delta_i}{2\alpha_i}$$

and

$$\overline{k} = \max_{1\leq i\leq d}\frac{2\alpha_i-\delta_i}{2\alpha_i}.$$

The rest arguments are similar to those given in the proof for Model 2. □

4. Transient and recurrent random walks

This section consists of three parts. In Section 4.1 the properties of the birth-and-death processes $BD(\gamma, d)$ are studied. In Section 4.2 the properties of random walks from Models 1, 2 and 3 are discussed and their extensions (Models 4 and 5) are studied. In Section 4.3 an application of the study to independent systems of null-recurrent birth-and-death processes is discussed.

4.1. **Birth-and-death processes.** We start from general birth-and-death processes with birth rates λ_n and death rates μ_n satisfying the properties $\lambda_n > \mu_n$ and $\lim_{n\to\infty} \lambda_n/\mu_n = 1$.

Lemma 4.1. *Let the birth and death rates* λ_n *and* μ_n *satisfy the properties* $\lambda_n > \mu_n$ *and* $\lim_{n\to\infty} \lambda_n/\mu_n = 1$. *Then, the birth-and-death process is transient if and only if*

$$\lim_{n\to\infty} \left(\frac{\lambda_n}{\mu_n}\right)^n = \mathrm{e}^z \tag{4.1}$$

is satisfied for $z > 1$ *and null-recurrent if and only if* (4.1) *is satisfied for* $z \leq 1$.

Proof. According to the known classification of birth-and-death processes [6], a birth-and-death process is null-recurrent if and only if $\sum_{n=1}^{\infty}\prod_{j=1}^{n}\lambda_{j-1}/\mu_j = \infty$ and $\sum_{n=1}^{\infty}\prod_{j=1}^{n}\mu_j/\lambda_j = \infty$ and transient if and only if $\sum_{n=1}^{\infty}\prod_{j=1}^{n}\lambda_{j-1}/\mu_j = \infty$ and $\sum_{n=1}^{\infty}\prod_{j=1}^{n}\mu_j/\lambda_j < \infty$. Apparently, the first condition $\sum_{n=1}^{\infty}\prod_{j=1}^{n}\lambda_{j-1}/\mu_j = \infty$ is always satisfied. Indeed, since $\lim_{n\to\infty}\lambda_n/\mu_n = 1$, one can assume that both sequences $\{\lambda_n\}$ and $\{\mu_n\}$ are properly normalized such that, starting from large N, both of them are less than and greater than some constants C_1 and C_2, respectively. Then, $\prod_{j=1}^{n}\lambda_{j-1}/\mu_j$ does not vanish as $n\to\infty$, and hence the required series diverges. Next, $\sum_{n=1}^{\infty}\prod_{j=1}^{n}\mu_j/\lambda_j < \infty$ is satisfied if and only if there exist constants C and $\epsilon > 0$ such that for n large enough and all $N > n$,

$$\prod_{j=1}^{N}\frac{\mu_j}{\lambda_j} \leq \frac{C}{N^{1+\epsilon}}. \tag{4.2}$$

In turn, $\sum_{n=1}^{\infty}\prod_{j=1}^{n}\mu_j/\lambda_j = \infty$ is satisfied if and only if for any $\epsilon > 0$ there exists n large enough such that for all $N > n$

$$\prod_{j=1}^{N}\frac{\mu_j}{\lambda_j} > \frac{1}{N^{1+\epsilon}}. \tag{4.3}$$

It is not difficult to show that (4.3) implies the asymptotic expansion

$$\frac{\mu_n}{\lambda_n} \asymp 1 - \frac{1}{zn} \tag{4.4}$$

for some $z \leq 1$. Indeed, if

$$\prod_{j=1}^{n}\frac{\mu_j}{\lambda_j} \asymp \frac{C}{n}$$

for some constant C and n increasing to infinity, then it is readily seen that we arrive at

$$\frac{\mu_n}{\lambda_n} \asymp 1 - \frac{1}{n}.$$

Hence, under condition (4.3) we obtain (4.4) with $z \leq 1$, and similarly, under condition (4.2) we obtain (4.4) with $z > 1$. The obtained expansions imply the corresponding limits in (4.1). The lemma is proved. □

Lemma 4.2. *The family of birth-and-death processes $BD(\gamma, d)$ is null recurrent for $d \leq 2$ and transient for $d \geq 3$.*

Proof. As $n \to \infty$, we obtain:

$$
\begin{aligned}
\frac{\lambda_n(\gamma,d)}{\mu_n(\gamma,d)} &= \frac{d\gamma^d\binom{n-1}{d-2} + (d-1)d\gamma^{d-1}\binom{n-1}{d-2} + d\gamma^d\binom{n-1}{d-1}}{(d-1)d\gamma^{d-1}\binom{n-1}{d-2} + d\gamma^d\binom{n-1}{d-1}}\left[1 + O\left(\frac{1}{n}\right)\right]\\
&= \frac{(d-1+\gamma)\binom{n-1}{d-2} + \gamma\binom{n-1}{d-1}}{(d-1)\binom{n-1}{d-2} + \gamma\binom{n-1}{d-1}}\left[1 + O\left(\frac{1}{n}\right)\right]\\
&= \frac{\frac{1}{\gamma}(d-1)(d-1+\gamma) + (n-d+1)}{\frac{1}{\gamma}(d-1)^2 + (n-d+1)}\left[1 + O\left(\frac{1}{n}\right)\right].
\end{aligned} \tag{4.5}
$$

Hence,

$$
\lim_{n\to\infty}\left(\frac{\lambda_n(\gamma,d)}{\mu_n(\gamma,d)}\right)^n = \mathrm{e}^{d-1}. \tag{4.6}
$$

Thus, by virtue of Lemma 4.1 we arrive at the conclusion that $BD(\gamma, d)$ is recurrent for $d \leq 2$ and transient for $d \geq 3$. The lemma is proved. □

4.2. **Families of random walks.**

4.2.1. *The results for Models 1, 2 and 3.*

Proposition 4.3. *The family of symmetric random walks $\{\mathbf{S}_t, \mathcal{A}, d\}$ is recurrent for $d \leq 2$ and transient for $d \geq 3$.*

Proof. It follows from Theorem 2.1 that the family of symmetric random walks is $(\mathcal{A}, d)$-conservative and associated with $BD(2, d)$. Hence, owing to Lemma 4.2, the family of symmetric random walks is recurrent for $d \leq 2$ and transient for $d \geq 3$. □

Proposition 4.4. *The family of random walks in Models 2 and 3 is recurrent for $d \leq 2$ and transient for $d \geq 3$.*

Proof. Indeed, according to Theorem 3.1 the families of random walks in Model 2 or 3 are semiconservative. The birth probabilities of the associated birth-and-death process are bounded by the two-sided inequalities of the birth probabilities of $BD(2\underline{k}, d)$ and $BD(2\overline{k}, d)$ processes. According to Lemma 4.2 both of these birth-and-death processes are recurrent for $d \leq 2$ and transient for $d \geq 3$, and hence, the associated families of random walks are recurrent for $d \leq 2$ and transient for $d \geq 3$. □

4.2.2. *An extended model.* In this section we consider a general random walk, which is an extension of the random walks in Models 1, 2 and 3.

Model 4. We consider the family of random walks defined by (1.1) and (1.2). Assume that $\mathbf{e}_t$ depends on the state $\mathbf{S}_{t-1}$ as follows. It takes values $\mathbf{1}_i$ or $(-\mathbf{1}_i)$ with probability $\tilde{\alpha}_i\big(S_{t-1}^{(i)}\big) \geq c > 0$, where $\tilde{\alpha}_i\big(S_{t-1}^{(i)}\big) \leq \alpha_i$ and $2\sum_{i=1}^d \alpha_i = 1$, and $\mathbf{e}_t$ takes value $\mathbf{0}$ with the complementary probability, $1 - 2\sum_{i=1}^d \tilde{\alpha}_i\big(S_{t-1}^{(i)}\big)$. The value $c < \min\{\alpha_1, \alpha_2, \ldots, \alpha_d\}$ is an arbitrarily small positive value.

Proposition 4.5. *The family of random walks in Model 4 is recurrent for $d \leq 2$ and transient for $d \geq 3$.*

Proof. The proof is based on coupling arguments. Consider first the following two auxiliary models. In the first model (call it Model A1), for all $n \geq 1$

$$\tilde{\alpha}_i(n) = \alpha_i, \quad \text{and} \quad 2\sum_{i=1}^{d}\alpha_i = 1,$$

and

$$\tilde{\alpha}_i(0) = c.$$

In the second model (call it Model A2)

$$\tilde{\alpha}_i(0) = \alpha_i, \quad \text{and} \quad 2\sum_{i=1}^{d}\alpha_i = 1,$$

and for all $n \geq 1$

$$\tilde{\alpha}_i(n) = c.$$

Apparently, Model A1 is a version of Model 3, and Model A2 is a version of Model 2. According to Proposition 4.4, both of them are recurrent for $d \leq 2$ and transient for $d \geq 3$. Then the coupling arguments enable us to conclude that the same is true for Model 4, and the statements of Proposition 4.5 follow. □

4.2.3. *Further extension of Model 4.* We start from an extension of Model 1 and then, on the basis of it, we provide further extension of Model 4.

Model B1. For random walks defined by (1.1) and (1.2), let the vector $\mathbf{e}_t$ be one of the $2d$ randomly chosen vectors $\{\pm\mathbf{1}_i, i = 1, 2, \ldots, d\}$ as follows. For $S_{t-1}^{(i)} \geq 0$, the probability that the vector $\mathbf{1}_i$ will be chosen is $\tilde{\alpha}_i\big(S_{t-1}^{(i)}\big) > 0$, and the probability that the vector $(-\mathbf{1}_i)$ will be chosen is $\tilde{\beta}_i\big(S_{t-1}^{(i)}\big) > 0$, where

$$\tilde{\alpha}_i\big(S_{t-1}^{(i)}\big) + \tilde{\beta}_i\big(S_{t-1}^{(i)}\big) = 2\alpha_i, \quad 2\sum_{i=1}^{d}\alpha_i = 1.$$

The further specifications of the probabilities $\tilde{\alpha}_i\big(S_{t-1}^{(i)}\big)$ and $\tilde{\beta}_i\big(S_{t-1}^{(i)}\big)$ are as follows. Generally, we assume

$$\tilde{\alpha}_i\big(S_{t-1}^{(i)}\big) = \tilde{\beta}_i\big(-S_{t-1}^{(i)}\big), \tag{4.7}$$

and if $\big|S_{t-1}^{(i)}\big| > M$, then

$$\tilde{\alpha}_i\big(S_{t-1}^{(i)}\big) = \tilde{\beta}_i\big(S_{t-1}^{(i)}\big) = \alpha_i. \tag{4.8}$$

Note also that according to (4.7), $\tilde{\alpha}_i(0) = \tilde{\beta}_i(0) = \alpha_i$.

Lemma 4.6. *The random walk in Model B1 is recurrent for $d \leq 2$ and transient for $d \geq 3$.*

Proof. To simplify the derivations and make the results of calculations observable, we prove this lemma for the particular random walk, assuming that if $1 \leq \big|S_{t-1}^{(i)}\big| \leq M$, then

$$\tilde{\alpha}_i\big(S_{t-1}^{(i)}\big) = \alpha_i^*, \quad \tilde{\beta}_i\big(S_{t-1}^{(i)}\big) = 2\alpha_i - \alpha_i^*.$$

In addition, we assume that $\alpha_i^*/(2\alpha_i-\alpha_i^*)=\rho$ is the same constant for all i. The made assumption does not logically change the proof. Then the random walk is modeled by the following queueing system.

Consider the following series of d state-dependent Markovian queueing systems. The arrival rate of each of d mutually independent queueing systems depends on queue-length as follows. If the ith system is empty, the arrival rate is $2\alpha_i$. Otherwise, is there is at least one customer in the system, then the arrival rate is α_i. The service rate depends on the queue-length as follows. If immediately before a moment of service began there are more than M customers in the system, then the service rate is α_i. Otherwise, it is β_i. The values of parameters β_i are scaled on the basis of the original proportion between α_i^* and $2\alpha_i-\alpha_i^*$.

The proof of this lemma is similar to that of the proof of Theorem 2.1. We choose N large enough that is greater than M, and consider first an ith queueing system with N waiting places. Using the notation similar to that in the proof of Theorem 2.1, for that queueing system from the Chapman-Kolmogorov equations we obtain $P_N^{(i)}(1)=2(\alpha_i/\beta_i)P_N^{(i)}(0)=2\rho P_N^{(i)}(0)$, $P_N^{(i)}(n+1)=\rho P_N(n)$ for $n=0,1,\ldots,M-1$, and $P_N^{(i)}(n+1)=P_N^{(i)}(n)$ for $n>M-1$, and then,

$$P_N(\mathbf{n})=\prod_{i=1}^{d}P_N^{(i)}\left(n^{(i)}\right).$$

Next, assuming that $\|\mathbf{n}\|<N$, we have

$$P_N\left[\mathcal{N}^+(n)\right]=\sum_{\mathbf{n}\in\mathcal{N}^+(n)}P_N(\mathbf{n}). \tag{4.9}$$

Similarly to (2.14), the explicit presentation for the probability $P_N\left[\mathcal{N}^+(n)\right]$ can be derived, first in the case $\alpha_1=\alpha_2=\ldots=\alpha_d\equiv\alpha$ and then in the general case. For large n and $N>n$ we will derive the expansion for $p_n(d)$ based on the following results. First, we take into account that for any two vectors $\mathbf{n}_1$ and $\mathbf{n}_2$, the components of which all are greater than M, we have

$$\frac{P_N(\mathbf{n}_2)}{P_N(\mathbf{n}_1)}=1.$$

If all components of vector $\mathbf{n}_2$ are greater than M, while there are d_0 zero components of vector $\mathbf{n}_1$, then

$$\frac{P_N(\mathbf{n}_2)}{P_N(\mathbf{n}_1)}\geq 2^{d_0}\rho^{(M+1)d_0},$$

where the equality is satisfied in the only case where all other $d-d_0$ components are greater than M. In addition, as $N\to\infty$, similarly to (2.12)

$$\Pi_\infty(\mathbf{n}_1,\mathbf{n}_2)=\lim_{N\to\infty}\frac{P_N(\mathbf{n}_2)}{P_N(\mathbf{n}_1)}. \tag{4.10}$$

Then, for sufficiently large n, (4.9) is evaluated by

$$P_N\left[\mathcal{N}^+(n)\right]\asymp\sum_{\left\{\mathbf{n}\in\mathcal{N}^+(n):\quad \min\{n^{(1)},n^{(2)},\ldots,n^{(d)}\}>M\right\}}P_N(\mathbf{n}), \tag{4.11}$$

since the contribution of the terms $P_N(\mathbf{n})$ with

$$\mathbf{n}\in\mathcal{N}^+(n)\setminus\left\{\mathbf{n}\in\mathcal{N}^+(n):\quad \min_{1\leq i\leq d}n^{(i)}>M\right\}$$

is negligible in (4.9) as $n \to \infty$. Hence, based on (4.11) and the earlier result in (2.16), for sufficiently large n and $N > n$ we have the estimate

$$p_n(d) \asymp \frac{C_0(2\rho^{M+1}, n, d) + C(2\rho^{M+1}, n, d)}{C_0(2\rho^{M+1}, n, d) + 2C(2\rho^{M+1}, n, d)},$$

where

$$C(\gamma, n, d) = \sum_{i=1}^{d} i\gamma^i \binom{d}{i}\binom{n-1}{i-1},$$

and

$$C_0(\gamma, n, d) = \gamma \sum_{i=1}^{d} (d-i)\gamma^i \binom{d}{i}\binom{n-1}{i-1}.$$

Hence, as $n \to \infty$, the probability $p_n(d)$ is asymptotically equivalent with the birth probability $p(n)$ in the birth-and-death process $BD(2\rho^{M+1}, d)$. Thus, according to Lemma 4.2 the random walk is recurrent for $d \leq 2$ and transient for $d \geq 3$. The proof is completed. □

Model 5. We consider the family of random walks defined by (1.1) and (1.2). Assume that $\mathbf{e}_t$ depends on the state $\mathbf{S}_{t-1}$ as follows. It takes value $\mathbf{1}_i$ with probability $\tilde{\alpha}_i\big(S_{t-1}^{(i)}\big) \geq c > 0$, value $(-\mathbf{1}_i)$ with probability $\tilde{\beta}_i\big(S_{t-1}^{(i)}\big) \geq c > 0$, where $\tilde{\alpha}_i\big(S_{t-1}^{(i)}\big) + \tilde{\beta}_i\big(S_{t-1}^{(i)}\big) \leq 2\alpha_i$, $2\sum_{i=1}^{d} \alpha_i = 1$, and $\mathbf{e}_t$ takes value $\mathbf{0}$ with the complementary probability $1 - \sum_{i=1}^{d}\big[\tilde{\alpha}_i\big(S_{t-1}^{(i)}\big) + \tilde{\beta}_i\big(S_{t-1}^{(i)}\big)\big]$. The value $c < \min\{\alpha_1, \alpha_2, \ldots, \alpha_d\}$ is an arbitrarily small positive value. The further specifications of the probabilities $\tilde{\alpha}_i\big(S_{t-1}^{(i)}\big)$ and $\tilde{\beta}_i\big(S_{t-1}^{(i)}\big)$ are as follows. Generally, we assume (4.7), and if $\big|S_{t-1}^{(i)}\big| > M$, then (4.8).

Theorem 4.7. *The random walk in Model 5 is recurrent for $d \leq 2$ and transient for $d \geq 3$.*

Proof. The construction of the proof is as follows. As Model B1 is an extension of Model 1, the similar extensions (called Models B2 and B3) can be constructed for Models 2 and 3, respectively, and the proof of Lemma 4.6 can be adapted to new Models B2 and B3 as well. Then, the statement of Theorem 4.7 is proved by the way that is used to prove Proposition 4.5 based on coupling arguments. □

Remark 4.8. Condition (4.7) that describes Model B1 is technical. It is used for reduction of the original random work of Model B1 to the reflected random walk, which in turn is described by the queueing system constructed in the proof. We reckon that the statement of Lemma 4.6 and Theorem 4.7 might be correct for the more general models that do not include this condition.

Remark 4.9. The condition $\tilde{\alpha}_i\big(S_{t-1}^{(i)}\big) + \tilde{\beta}_i\big(S_{t-1}^{(i)}\big) \leq 2\alpha_i$ for Model 5 as well as the similar condition $\tilde{\alpha}_i\big(S_{t-1}^{(i)}\big) \leq \alpha_i$ for Model 4 are important. They guarantee that the components $S_t^{(i)}$, $i = 1, 2, \ldots, d$ in the corresponding random walks $\mathbf{S}_t$ are independent.

4.3. **Further examples of recurrent and transient random walk.** The random walks that are described by Model 5 are characterized as follows. Let $\breve{\mathbf{S}}_t = \big(\breve{S}_t^{(1)}, \breve{S}_t^{(2)}, \ldots, \breve{S}_t^{(d)}\big)$ be the reflected random walks. Under the assumption that the random walks stay in each of their states for an exponentially distributed time, the components $\breve{S}_t^{(i)}$, $i = 1, 2, \ldots, d$ are thought as the null-recurrent birth and death processes with the birth rates λ_n and death rates μ_n satisfying the property $\lambda_n = \mu_n$ for $n \geq M + 1$. In this section we discuss more general situation of the system of d independent null-recurrent birth-and-death processes.

Example 4.10. Let $S_t^{(1)}$ and $S_t^{(2)}$ be two null-recurrent independent birth-and-death processes, and let $S_0^{(1)} = S_0^{(2)} = 0$. For simplicity of our analysis, assume that the birth-and-death processes $S_t^{(1)}$ and $S_t^{(2)}$ are identically distributed. That is both of them are specified by the same birth rates L_n and death rates M_n.

Let $\tau = \inf\{t > 0 : S_{t+h}^{(1)} + S_{t+h}^{(2)} = 0\}$, where $h > 0$ is an arbitrary constant.

Theorem 4.11. *Assume that*

$$\lim_{n\to\infty} \left(\frac{L_n}{M_n}\right)^n > 1.$$

Then $\mathsf{P}\{\tau < \infty\} < 1$.

Proof. The proof is similar to that of Theorem 2.1 and based on asymptotic analysis similar to that provided in this section to prove Lemma 4.2. First, taking N large, we consider two independent Markovian queueing systems with N waiting places. For simplicity, we assume that service times are identically distributed with rate 1, and interarrival times are identically distributed with rate $1 + c/N$, $c > 0$ is some positive constant. The following arguments of the proof are similar to those given in the proof of Theorem 2.1, where we derive the asymptotic expression for $p_n(2)$ as $N \to \infty$, and then in the proof of Lemma 4.2, where we derive $\lim_{n\to\infty} \big[\lambda_n(\gamma, 2)/\mu_n(\gamma, 2)\big]^n$. Note, that the asymptotic behaviour in (4.5) does not depend on γ. Denote

$$p_n = \lim_{t\to\infty} \mathsf{P}\left\{S_{t+1}^{(1)} + S_{t+1}^{(2)} = n + 1 \mid S_t^{(1)} + S_t^{(2)} = n\right\}.$$

Taking in account that for any $0 < \kappa < 1$ and $n \to \infty$

$$\left(\frac{L_{\lfloor \kappa n\rfloor}}{M_{\lfloor \kappa n\rfloor}} \cdot \frac{L_{n-\lfloor \kappa n\rfloor}}{M_{n-\lfloor \kappa n\rfloor}}\right)^n \asymp \left(\frac{L_n}{M_n}\right)^n$$

($\lfloor a\rfloor$ denotes the integer part of a), as $n \to \infty$ we obtain

$$\left(\frac{p_n}{1 - p_n}\right)^n \asymp \left(\frac{\lambda_n(1, 2)}{\mu_n(1, 2)} \cdot \frac{L_n}{M_n}\right)^n,$$

and hence,

$$\lim_{n\to\infty} \left(\frac{p_n}{1 - p_n}\right)^n = \mathrm{e}^{c+1}.$$

Since $c > 0$, then according to Lemma 4.1 we obtain $\mathsf{P}\{\tau < \infty\} < 1$. $\square$

Example 4.12. Let $S_t^{(1)}$, $S_t^{(2)}, \ldots, S_t^{(d)}$ $(d \geq 3)$, be independent, identically distributed, null-recurrent birth-and-death processes. Let

$$\tau = \inf \left\{ t > 0 : \sum_{i=1}^{d} S_{t+h}^{(i)} = 0 \right\},$$

where $h > 0$ is an arbitrary constant. Denote the birth and death rates by L_n and M_n, respectively.

Theorem 4.13. *Assume that*

$$\lim_{n\to\infty} \left(\frac{L_n}{M_n} \right)^n \leq \mathrm{e}^{2-d}. \tag{4.12}$$

Then $\mathsf{P}\{\tau < \infty\} = 1$.

Proof. Using the similar arguments as in the proof of Theorem 4.11, we have as follows. Let

$$p_n = \lim_{t\to\infty} \mathsf{P} \left\{ \sum_{i=1}^{d} S_{t+1}^{(i)} = n + 1 \mid \sum_{i=1}^{d} S_t^{(i)} = n \right\}.$$

As $n \to \infty$ we obtain

$$\left(\frac{p_n}{1 - p_n} \right)^n \asymp \left[\frac{\lambda_n(1,2)}{\mu_n(1,2)} \cdot \frac{L_n}{M_n} \right]^n.$$

Hence,

$$\lim_{n\to\infty} \left(\frac{p_n}{1 - p_n} \right)^n \leq \mathrm{e}^{d-1}\mathrm{e}^{2-d} = \mathrm{e}.$$

Then, the statement of the theorem follows from Lemma 4.1. □

5. Discussion and concluding remarks

In the present paper, we gave a new classification of multidimensional random walks. Based on that classification, we established new results on the behaviour of random walks. The main techniques used in the paper are reduction to birth-and-death processes, asymptotic analysis and coupling arguments. The concepts of conservative and semiconservative random walks are of independent interest. The principally new results of the paper include the analysis of Examples 4.10 and 4.12 resulted in the proof of Theorems 4.11 and 4.13. The statement of Proposition 4.3 was previously covered by the results in Chung and Fuchs [1] (see also [2]) and Foster and Good [5]. A version of the proof of Chung and Fuchs theorem is presented in Durrett [4]. Specifically, Theorem 4.2.8 on page 166 and Theorem 4.2.13 on page 170 together claim that any unbiased random walk in $\mathbb{R}^d$ having increments in the domain of attraction of a Gaussian distribution is transient if and only if $d \geq 3$. The classes of random walks in [1] and [5], however, do not cover state-dependent random walks considered in Models 2, 3, 4 and 5. MacPhee and Manshikov [10] showed that a nonzero drift of a random walk on a lower-dimensional subspace is sufficient in order to change the recurrence classification. The method of Lyapunov functions that is used by Lamperti [7] provides intuition for the phase transition. In its simplest version, the idea is to consider the process $\varphi_t = \|\mathbf{S}_t\| = \left[\sum_{i=1}^{d} \left(S_t^{(i)} \right)^2 \right]^{1/2}$, the recurrence or transience of which is determined by comparing

$$\mathsf{E}\{\varphi_{t+1} - \varphi_t | \mathbf{S}_t = \mathbf{x}\}$$

and

$$\mathsf{E}\{(\varphi_{t+1} - \varphi_t)^2 | \mathbf{S}_t = \mathbf{x}\}.$$

It would be interesting to investigate the applicability of the method in [7] to the models under consideration here.

The results by Doyle and Snell [3] also concern state-dependent random walks similar to those described by Model B1 in the framework of electric networks theory. To be specific, we refer the arXiv version of the book, where the relevant results are in Section 2.4 "Random walks on more general infinite networks", page 101. The formulation and proof of the basic theorem is given on page 102. The formulated theorem violates the conditions mentioned in Remark 4.9. Unfortunately, we could not follow the proof of that theorem.

Acknowledgement

The author thanks the anonymous referee for valuable comments leading to substantial improvement of the paper.

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