A new simple proof of Kummer’s test


1. INTRODUCTION. Let

\[ \sum_{n=0}^{\infty} a_n \]  

(1)

be a positive series. In 1835, Ernst Kummer [7] built a universal ratio test for convergence or divergence of positive series. The original version of the test was restrictive, and its proof was complicated. After more than fifty years since its publication Stoltz provided the clearer formulation and proof that has been well-accepted and appeared in the textbooks (see e. g. [6, page 311]) and well-known in our days. Being more general than many of existing particular tests such as d’Alembert test, Raabe’s test, Bertrand’s test and Gauss’s test, it has been of the constant attention in the literature. It serves as a source of new particular tests (e. g. [1]), has applications in the theory of difference equations (e. g. [3]), and there is a variety of new proofs of this test in the literature (see [2, 6, 8, 9, 10]).

It was shown in [10] that Kummer’s test covers all positive series, and the convergence or divergence of this test can be formulated in the form of the necessary and sufficient conditions. A short proof for the necessary and sufficient condition of the only convergence of (1) can be found in [9].

In the present note, we provide another short proof of Kummer’s test in the formulation given in [10]. The advantage of our approach is that it is based on a known Hardy-Littlewood Tauberian theorem, and the proof is mostly free of algebraic derivations.

2. KUMMER’S TEST AND ITS PROOF. We provide a new proof of the theorem given in [10]. The formulation of that theorem is as follows.

Theorem 1. We have the following two claims.

(i) Series (1) converges if and only if there exists a positive sequence \( \zeta_n, n = 0, 1, \ldots \), such that \( \zeta_n (a_n / a_{n+1}) - \zeta_{n+1} \geq c > 0 \).

(ii) Series (1) diverges if and only if there exists a positive sequence \( \zeta_n, n = 0, 1, \ldots \), such that \( \zeta_n (a_n / a_{n+1}) - \zeta_{n+1} \leq 0 \) and \( \sum_{n=0}^{\infty} 1 / \zeta_n = \infty \).

Proof. Recall the Hardy-Littlewood Tauberian theorem that will be used in this proof (see [4, 5]).

Lemma. Let the series \( \sum_{j=0}^{\infty} a_j x^j \) converge for \(-1 < x < 1\), and suppose that there exists \( \gamma > 0 \) such that \( \lim_{x \uparrow 1} (1 - x)^\gamma \sum_{n=0}^{\infty} a_j x^j = A \). Suppose also that \( a_j \geq 0 \). Then, as \( N \to \infty \), we have \( \sum_{j=0}^{N} a_j = (A / \Gamma(1 + \gamma)) N^\gamma (1 + o(1)) \), where \( \Gamma(x) \) is Euler’s Gamma-function.

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Proof of claim (i). Denoting \( c_n = \zeta_n(a_n/a_{n+1}) - \zeta_{n+1} \), write \( c_na_{n+1} = z_n - z_{n+1} \), where \( z_n = \zeta_na_n \). Now, for \( x \leq 1 \) we use generating functions. Let \( Z(x) = \sum_{n=0}^{\infty} z_n x^n \), and \( A(x) = \sum_{n=0}^{\infty} c_na_{n+1} x^{n+1} \). We have the following relationship
\[
a_0\zeta_0 - A(x) = (1 - x)Z(x).
\] (2)

Proof of ‘only if’ part. If series (1) is convergent, then the sequence \( \zeta_n \) can be chosen as follows. Suppose that the sequence \( c_n \) is positive and bounded. Then Abel’s theorem applies, and we have \( \lim_{x\to 1^-} A(x) = \sum_{n=0}^{\infty} c_na_{n+1} = s \). So, \( \zeta_0 \) can be chosen such that \( a_0\zeta_0 - s > 0 \). Then, according to Lemma, for large \( N \) we have \( \sum_{n=0}^{N} z_n = (a_0\zeta_0 - s)N(1 + o(1)) \), so the sequence of \( \zeta_n \) exists. To this end, we are to show that the sequence \( c_n \) is indeed bounded. Here we use an argument of [9]. For large \( n \) we have \( z_{n+1} = z_n - c_na_{n+1} = z_n - (c - c + c)a_{n+1} \), where \( c \) is the lower bound of the sequence \( c_n \). So, \( z_{n+1}' = z_n + (c - c)a_{n+1} = z_n - ca_{n+1} \). By this, we arrive at the new sequence \( z_n' \) with keeping all previous arguments the same. Hence, the relation \( \zeta_n(a_n/a_{n+1}) - \zeta_{n+1} = c > 0 \) implies the convergence of (1).

Proof of ‘if’ part. If we assume that the relation \( \zeta_n(a_n/a_{n+1}) - \zeta_{n+1} = c > 0 \) is true, we arrive at the claim that (1) converges, since the left-hand side of (2) must be positive and therefore finite.

Proof of claim (ii). We are to prove the following equivalent proposition.

Proposition. Assume that \( \zeta_n(a_n/a_{n+1}) - \zeta_{n+1} \leq c \leq 0 \). Then series (1) converges if and only if \( \sum_{n=0}^{\infty} 1/\zeta_n < \infty \).

Proof of ‘only if’ part. Assume that (1) converges. If \( c_n \) is bounded, then using Lemma for large \( N \) we arrive at the same estimate as in the proof of claim (i): \( \sum_{n=0}^{N} z_n = (a_0\zeta_0 - s)N(1 + o(1)) \). Taking into account that the sequence \( z_n \) is increasing, it must converge to the limit. So, for large \( n \) we have \( \zeta_n = O(1/a_n) \), which means that the convergence of (1) implies the convergence of \( \sum_{n=0}^{\infty} 1/\zeta_n \). To this end, we are to show that the sequence \( c_n \) is indeed bounded. As in the proof of claim (i), for large \( n \) we have \( z_{n+1} = z_n - c_na_{n+1} = z_n - (c - c + c)a_{n+1} \), where \( c \) is the upper bound of the sequence \( c_n \). So, \( z_{n+1}' = z_n + (c - c)a_{n+1} = z_n - ca_{n+1} \). By this, we arrive at the new sequence \( z_n' \) with keeping all previous arguments the same.

Proof of ‘if’ part. If \( \sum_{n=0}^{\infty} 1/\zeta_n < \infty \), and \( \zeta_{n+1}a_{n+1} - \zeta_n a_n = ca_{n+1} \) for large \( n \), then \( a_n \leq C/\zeta_n \) for some constant \( C \), and (1) must converge.

REFERENCES

